$$\frac{dx}{dt} = u_1, \quad \frac{dy}{dt} = u_2(x^2 + y^2), \qquad \sqrt{(u_1^2 + u_2^2)} \leqslant 1$$

It is easy to show that the point $x = 0,\ y = 0$ is not an interior point of the set $F: \{|x| \leqslant 1,\ y = 0\}$, but that the Bellman function satisfies the Lipschitz condition in the neighborhood of every point of space except at the origin.

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OPTIMAL CONTROL OF SYSTEMS WITH LAG BY SUITABLE CHOICE OF THE INITIAL CONDITIONS

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The problem of bringing a system with lag to a specified position by suitable choice of the initial conditions is considered. The conditions of solvability of this problem are formulated in terms of the coefficients of the eqautions.

For simplicity we shall consider equations with constant coefficients defined in the n-dimensional Euclidean space E_n,

$$x^{\cdot}(t) = \sum_{i=1}^{m} x(t - h_i) B_i + x^{\cdot}(t - h) B_0 + f(t), \qquad t > 0 \tag{1}$$

where $x(t)$ is an n-dimensional vector. We assume that the coefficients of Eq. (1) satisfy the following Conditions (A): the lag constants h_i are such that $h_m > h_{m-1} \geqslant \ldots \ldots \geqslant h_1 \geqslant 0$, that the constant $h > 0$, that the continuous function $f(t)$ assumes values from the space E_n, and finally, that $B_i,\ i = 0, \ldots, m$, are square $n \times n$ matrices with constant elements. We also stipulate that all the vectors from E_n ocurring below are to be regarded as vector rows; we denote the jth coordinate of a vector from E_n by the same letter as the vector with the subscript j. For example, the vector $x(t) = (x_1(t), \ldots, x_n(t))$.

The solution $x(t)$ of Eq. (1) for $t > 0$ is subject to the initial conditions

$$x(0) = x_0,\; x(t) = \varphi(t) \quad \text{for } -h_m \leqslant t < 0,$$
$$x^{\cdot}(t) = \psi(t) \quad \text{for } -h \leqslant t \leqslant 0 \tag{2}$$

Here the vector $x_0 \in E_n$ and the initial functions $\varphi(t)$, $\psi(t)$ with values from the space E_n are given; the functions $\varphi(t)$, $\psi(t)$ are Borel-measurable, the function $\psi(t)$ is bounded, and $\|\varphi\| < \infty$, where

$$\|\varphi\| = \Big(\int_{-h_m}^{0} \sum_{i=1}^{n} \varphi_i^2(s)\, ds \Big)^{1/2} \tag{3}$$

Integrating Eq. (1) successively over the intervals $[kh, (k+1)h]$, $k = 0, 1, \ldots$, we see that fulfillment of Conditions (A) mean that there exists a unique pair of functions $x(t)$, $x^{\cdot}(t)$ which satisfies initial conditions (2) for $t \leqslant 0$ and Eq. (1) for almost all $t > 0$ (in the Lebesgue measure); moreover, $x^{\cdot}(t)$ is bounded and Lebesgue-integrable over any finite interval, and

$$x(t) = x(0) + \int_0^t x^{\cdot}(s)\, ds \quad \text{for } t \geqslant 0$$

We shall denote the solution of Eq. (1) under initial conditions (2) by $x(t, x_0, \varphi, \psi)$.

Problem 1. Let the vectors x_0, $x_1 \in E_n$, the initial function $\psi(t)$, and the instant $T > 0$ be given. We are to determine the initial function $\varphi(\theta)$ (the argument θ varies in the range $-h_m \leqslant \theta \leqslant 0$) with the smallest possible norm (3) in such a way that $x(T, x_0, \varphi, \psi) = x_1$.

We note that by virtue of [1], Sect. 12 the results obtained below are also valid for norms of the initial functions defined by any of the formulas

$$\operatorname{vrai\,max} \sum_{i=1}^{n} |\varphi_i(t)|, \quad \sup \sum_{i=1}^{n} \varphi_i^2(t), \quad -h_m \leqslant t \leqslant 0$$

However, to be specific, we shall limit ourselves to the norm defined by Eq. (3).

Let us find the Cauchy formula expressing the solution $x(t)$ of Eq. (1) as a function of initial conditions (2) and the inhomogeneities $f(t)$. To this end we introduce the function $\chi(s)$ which is equal to zero for $s = 0$ and to unity for $s > 0$, and set $\alpha(t) = [t/h] + \chi(t - h[t/h])$, where $[t]$ is the whole part of the number t. We can determine the $(n \times n)$-matrix $y(s, t)$ by means of the relations

$$\frac{\partial y(s,t)}{\partial s} = -\sum_{i=1}^{m} \sum_{j=0}^{\alpha(t)-1} B_i B_0^{\,j} y(s + h_i + jh, t) \quad (0 \leqslant s < t)$$
$$y(t, t) = I, \quad y(s, t) \equiv 0 \quad (s > t) \tag{4}$$

where I is an identity matrix and B^j denotes the jth power of the matrix B.

The solution $x(t)$ of problem (1), (2) can then be written as

$$x(t) = x_0 y(0, t) + \sum_{i=1}^{m} \int_{-h_i}^{0} \varphi(s) B_i \sum_{j=0}^{\alpha(t)-1} B_0^{\,j} y(s + h_i + jh, t)\, ds +$$
$$+ \int_{-h}^{0} \psi(s) \sum_{j=1}^{\alpha(t)} B_0^{\,j} y(s + jh, t)\, ds + \sum_{j=0}^{\alpha(t)-1} \int_0^t f(s) B_0^{\,j} y(s + jh, t)\, ds \tag{5}$$

To prove Eq. (5) we multiply both sides of Eq. (1) by the matrix $y\ (s,\ t)$ on the right and integrate the result from zero to t. This yields

$$\int_0^t \Big[x^{\cdot}(s) - \sum_{i=1}^{m} x(s-h_i) B_i - f(s)\Big] y(s,\ t)\, ds = \int_0^t x^{\cdot}(s-h) B_0 y(s,\ t)\, ds$$

Recalling (2), we integrate by parts in the left side of this relation. This is possible by virtue of the aforementioned absolute continuity of $x\ (t)$. Further, on the basis of (2) we obtain

$$\int_0^t x^{\cdot}(s-h) B_0 y(s,\ t)\, ds = \int_{-h}^{0} \psi(s) B_0 y(s+h,\ t)\, ds + \int_0^t x^{\cdot}(s) B_0 y(s+h,\ t)\, ds$$

We replace the derivative $x^{\cdot}\ (t)$ by its expression given by the right side of Eq. (1) and transform the result as above. Continuing with this iterative process, which consists in the successive replacement of the derivative $x^{\cdot}\ (t)$ in accordance with Eq. (1) and contains only a finite number of steps, namely $\alpha\ (t)$, we see that the expression (5) is valid.

Setting $t = T$ in formula (5) and denoting the value of the function $\alpha\ (t)$ at the point T by α, we conclude that if Problem 1 had a solution, then the following equation would hold:

$$\beta = \sum_{i=1}^{m} \int_{-h_i}^{0} \varphi(s) B_i \sum_{j=0}^{\alpha-1} B_0^{\ j} y(s+h_i+jh,\ T)\, ds \qquad (6)$$

Here the vector β is given by the expression

$$\beta = x_1 - x_0 y(0,\ T) - \int_{-h}^{0} \psi(s) \sum_{j=1}^{\alpha} B_0^{\ j} y(s+jh,\ T)\, ds - \sum_{j=0}^{\alpha-1} \int_0^T f(s) B_0^{\ j} y(s+jh,\ T)\, ds \qquad (7)$$

If the vector $\beta = 0$, then Problem 1 always has a solution, namely $\varphi\ (\theta) \equiv 0$, so that from now on we confine our attention to the case $\beta \neq 0$. We set

$$r_i(T,\ s) = \sum_{j=0}^{\alpha-1} B_0^{\ j} y(s+h_i+jh,\ T), \qquad -h_i \leqslant s \leqslant 0$$

$$r_i(T,\ s) = 0, \qquad -h_m \leqslant s < -h_i$$

and rewrite relation (6) as

$$\beta = \int_{-h_m}^{0} \varphi(s) \sum_{i=1}^{m} B_i r_i(T,\ s)\, ds \qquad (8)$$

Equation (8) implies that Problem 1 for Eq. (1) has been reduced to the problem of moments (e. g. see [1], Sect. 16).

Let us formulate the solvability conditions for Problem 1 following the results of monograph [1]. To this end we must define the set L of vectors $l \in E_n$ such that

$$\beta_l' = l_1\beta_1 + \ldots + l_n\beta_n = 1$$

(here and below the prime denotes transposition) and the set $\mathbf{P}$ of functions of the form

$$l_1\gamma_1(\theta)+\ldots+l_n\gamma_n\quad(\theta)\,(-h_m\leqslant\theta\leqslant 0,\, l\in L)$$

where γ_i (θ) are the columns of the matrix $B_1r_1\,(T,\theta)+\ .\ .\ .\ +B_mr_m\,(T,\,\theta)$.

Lemma 1. Let the coefficients of Eq. (1) satisfy Conditions (A) and let the vector $\beta\neq 0$. Then

1) the function φ (θ) is the solution of Eq. (8) if and only if

$$\int_{-h_m}^{0} g\,(s)\,\varphi\,(s)\,ds=1,\qquad g\,(\theta)\in P$$

2) Problem 1 has a solution if and only if

$$\rho_0=\min_{g(\theta)\in P}\|\,g\,\|>0\tag{9}$$

(here $\|\,g\,\|$ is defined by Eq. (3)).

3) if $\rho_0>0$ and if minimum (9) is attained on the function g_0 (θ), then the norm of the function which solves Problem 1 cannot be made smaller than $\rho_0{}^{-1}$, and the optimal initial function φ_0 (θ) such that $\|\,\varphi_0\,\|=\rho_0{}^{-1}$ has the following maximum property:

$$\int_{-h_m}^{0}\varphi\ (s)\,g_0{}'\,(s)\,ds=\max_{\varphi(\theta),\,\|\varphi\|=\rho_0^{-1}}\int_{-h_m}^{0}\varphi\,(s)\,g_0{}'\,(s)\,ds$$

The validity of Lemma 1 follows directly from the solution of the problem of moments ([1], Sect. 16).

Now let us derive the conditions in terms of coefficients which will enable us to draw conclusions concerning the existence of a solution of Problem 1 for n-dimensional equations

$$x^{\cdot}(t)=x\,(t)\,B_1+x\,(t-h)\,B_2+x^{\cdot}\,(t-h)\,B_0\qquad(t>0)\tag{10}$$

Theorem 1. If $h>0$, if the matrices B_2 and $B_2+B_0B_1$ are nondegenerate, and if $B_0B_i=B_iB_0$, $i=1,2$, then Problem 1 for Eq. (10) is solvable for all vectors x_0, x_1.

Proof. Let the variable τ vary in the range $0\leqslant\tau\leqslant h$. Then, by virtue of Lemma 1, we need merely show that the columns of the matrix

$$B_2\sum_{j=0}^{\alpha-1}B_0{}^j\,y\,(\tau+jh,\,T)$$

are linearly independent. Let us assume that the opposite is true, i. e. that there exists a nonzero vector c for which

$$B_2\sum_{j=0}^{\alpha-1}B_0{}^j\,y\,(\tau+jh,\,T)\,c'\equiv 0\qquad(0<\tau<h)$$

This and the conditions of Theorem 1 imply the identity

$$\sum_{j=0}^{\alpha-1}B_0{}^j\,y\,(\tau+jh,\,T)\,c'=0\tag{11}$$

Moreover, making use of relations (4) for the fundamental matrix $y(s,\ t)$ of system (10), we see the validity of the following chain of equations:

$$\frac{\partial y\,(\tau+ih,\,T)}{\partial\tau}+B_1\sum_{j=0}^{\alpha-i-1}B_0{}^j\,y\,(\tau+(i+j)\,h,T)=-B_2\sum_{j=1}^{\alpha-i}B_0^{j-1}\,y\,(\tau+(i+j)\,h,\,T)\tag{12}$$

Here the integer i runs through the values from 0 to $\alpha-1$.

Multiplying the ith equation of (12) by $B_0{}^i$ on the left and by c' on the right and summing the result over i from 0 to $\alpha-1$, we find by virtue of (11) that

$$\Big(B_1 \sum_{i=1}^{\alpha-1} \sum_{j=0}^{\alpha-i-1} B_0^{i+j} y(\tau+(i+j)h, T) + B_2 \sum_{i=0}^{\alpha-1} \sum_{j=1}^{\alpha-i} B_0^{i+j-1} y(\tau+(i+j)h, T)\Big) c' = 0 \tag{13}$$

But

$$\sum_{i=1}^{\alpha-1} \sum_{j=0}^{\alpha-i-1} B_0^{i+j} y(\tau+(i+j)h, T) = \sum_{i=1}^{\alpha-1} \sum_{j=i}^{\alpha-1} B_0^{j} y(\tau+jh, T) = \sum_{j=1}^{\alpha-1} jB_0^{j} y(\tau+jh, T) \tag{14}$$

and similarly, with allowance for (4),

$$\sum_{i=0}^{\alpha-1} \sum_{j=1}^{\alpha-i} B_0^{i+j-1} y(\tau+(i+j)h, T) = \sum_{i=0}^{\alpha-2} \sum_{j=i+1}^{\alpha-1} B_0^{j-1} y(\tau+jh, T) = \sum_{j=1}^{\alpha-1} jB_0^{j-1} y(\tau+jh, T)$$

This and (13), (14) give us the equation

$$(B_2+B_1B_0) \sum_{j=1}^{\alpha-1} jB_0^{j-1} y(\tau+jh, T) c' = 0$$

Hence,

$$\sum_{i=1}^{\alpha-1} iB_0^{i-1} y(\tau+ih, T) c' = 0, \qquad 0 \leqslant \tau \leqslant h \tag{15}$$

Let us add to identity (15) system of equations (12), with the index i in this case running through values from 1 to $\alpha-1$. Multiplying the ith equation of (12) by iB_0^{i-1} on the left and by c' on the right and summing the result from 1 to $\alpha-1$, we arrive, as with (13)–(15), at the formula

$$\sum_{i=2}^{\alpha-1} B_0^{i-2} y(\tau+ih, T) \frac{i(i-1)}{2} c' = 0 \tag{16}$$

Continuing the process used to derive (15), (16), we infer on the basis of mathematical induction and from the conditions of Theorem 1 that the result of the kth step takes the form of the relations

$$(B_2+B_1B_0) \sum_{i=k}^{\alpha-1} \gamma_{ki} B_0^{i-k} y(\tau+ih, T) c' = 0, \qquad 0 \leqslant \tau \leqslant h \tag{17}$$

where the numbers γ_{kj} are equal to zero for $k > j$, and are given by the relations

$$\gamma_{0j} \equiv 1, \qquad \gamma_{kj} = \sum_{i=k}^{j} \gamma_{k-1,\, i-1}, \quad k \geqslant 1$$

for $k \leqslant j$

For $i \geqslant 0$ all the $\gamma_{ii} = 1$; hence, setting $k = \alpha-1$ in formula (17), we find in accordance with (4) that $Ic' = 0$. However, this is not possible, as it contradicts the above assumption whereby $cc' \neq 0$. This contradiction proves Theorem 1.

Note 1. In proving Theorem 1 we showed that the columns of the matrix

$$r(\tau) = B_2 \sum_{j=0}^{\alpha-1} B_0^{j} y(\tau+jh, t) \qquad (-h \leqslant \tau \leqslant 0)$$

are linearly independent. This means that the matrix

$$G = \int_{-h}^{0} r'(s) r(s)\, ds \tag{18}$$

is nondegenerate. Hence, applying the method of undetermined Lagrange multipliers ([1], Sect. 18), we find that under the conditions of Theorem 1 the optimal initial function $\varphi(\tau)$ which solves Problem 1 for Eq. (10) is given by

$$\varphi(\tau) = \beta G^{-1} r'(\tau)$$

Theorem 2. Let $B_2 B_i = B_i B_2$, $i = 0, 1$, and let the constant $h > 0$. The nondegeneracy of the matrix B_2 then follows from the solvability of Problem 1 for Eq. (10) for all vectors $x_0, x_1 \in E_n$.

Proof. By virtue of Lemma 1, Problem 1 for Eq. (10) is solvable for all vectors x_0 x_1 only if (9) is fulfilled. Now let us suppose that the rank of the matrix B_2 is $m < n$ and show that condition (9) is violated in this case. Since, by virtue of formula (7), we can choose suitable values x_0, x_1 for any specified vector $\beta \in E_n$, we choose a vector β not contained in the space generated by the rows of the matrix B_2 (of dimension $m < n$) and a nonzero vector $l_0 \in L$ such that

$$\beta l'_0 = 1, \qquad B_2 l'_0 = 0$$

From this and (4) we obtain the equation

$$B_2 y(s, T)\, l'_0 \equiv 0 \quad \text{for } s \geqslant T \tag{19}$$

Multiplying Eq. (4) by B_2 on the left and by l'_0 on the right, we find that for $s < T$ the function $r(s) = B_2 y(s, T)\, l'_0$ is the solution of the equation

$$r^{\cdot}(s) = -\sum_{j=0}^{\alpha-1} [B_1 B_0^j r(s + jh) + B_2 B_0^j r(s + h + jh)]$$

under zero initial conditions (19). This means that $B_2 y(s, T)\, l'_0 \equiv 0$ for all $s < T$, which is impossible, since it contradicts Eq. (9). Theorem 2 has been proved.

Theorem 3. Let $B_i B_0 = B_0 B_i$, $i = 1, 2$, and let the constant $h > 0$. The solvability of Problem 1 for Eq. (10) for all x_0, x_1 then implies that the rank of the matrix

$$K = \{B_2, B_2 B_1, \ldots, B_2 B_1^{n-1}, B_2 + B_1 B_0, (B_2 + B_1 B_0) B_1, \ldots, (B_2 + B_1 B_0) B_1^{n-1}\}$$

is equal to the dimension n of system (10).

Proof. Let us suppose that the rank of the matrix K is $m < n$ and show that equality (9) (which is, by virtue of Lemma 1, the necessary condition of solvability of Problem 1 for Eq. (10)) is violated in this case. As in our proof of Theorem 2, the latter assumption implies the existence of a nonzero vector $l_0 \in L$ such that

$$\beta l_0' = 1, \qquad B_2 B_1^i l_0' = 0, \quad (B_2 + B_1 B_0) B_1^i l_0' = 0, \; i = 0, \ldots, n-1$$

Hence (see [1], p. 139), and for all integers $i \geqslant 0$ we have

$$B_2 B_1^i l_0' = 0, \qquad (B_2 + B_1 B_0) B_1^i l_0' = 0 \tag{20}$$

Let us consider the analytic function $r(s) = B_2 e^{B_1(T-s)} l'_0$ of the variable s. By virtue of (20) we have the equations

$$\left.\frac{d^i r(s)}{ds}\right|_{s=T} = B_2 B_1^i l_0' (-1)^i = 0, \qquad i = 0, 1, \ldots$$

from which we see that all the derivatives of the analytic function $r(s)$ are equal to zero for $s = T$. This means (e.g. see [2]) that $r(s) \equiv 0$ for all s. In similar fashion we can prove that the identity

$$r_1(s) = (B_2 + B_0B_1)\, e^{B_1(T-s)}\, l_0' \equiv 0. \tag{21}$$

holds for all s

By virtue of (4) the functions $r(s)$ and $r_1(s)$ coincide on the segment $T - h \leqslant s \leqslant T$ with the functions $B_2 y(s, T)\, l'_0$ and $(B_2 + B_0B_1)y(s, T)\, l'_0$; this means that for $T - h \leqslant \leqslant s \leqslant T$ we have

$$B_2 y(s,T)\, l_0' = (B_2 + B_0B_1)\, y(s,T)\, l_0' = 0 \tag{22}$$

Further, with allowance for (4) we find that

$$y(s,T) = e^{B_1(T-s)} - \int_{T-h}^{s} e^{B_1(t-s)}(B_1B_0 + B_2)\, y(t + h,T)\, dt \tag{23}$$

for $T - 2h \leqslant s \leqslant T - h$.

This and formulas (21), (22) imply the relations

$$y(s,T)\, l_0' = e^{B_1(T-s)}\, l_0', \qquad T - 2h \leqslant s \leqslant T - h \tag{24}$$

which, by virtue of the equations $r(s) = r_1(s) = 0$, enable us to infer that identities (22) are also valid for $s \in [T - 2h, T]$. Now let us suppose that identities (22) hold for $T - kh \leqslant s \leqslant T$ (the integer $k \geqslant 2$). Then, since by (4) and the conditions of Theorem 3 we have

$$y(T - kh,T) = e^{kB_1h} + V[(B_2 + B_0B_1)\, y(\tau_1\, T)]$$

where $V(Z(\tau_1))$ is a linear functional defined on the functions $Z(\tau_1)$ of the argument $\tau_1 \in [T - kh, T]$, it follows that formulas (22), as (23), (24), are also valid in the interval $[T - (k + 1)\, h, T]$. By mathematical induction we infer from this that Eqs. (22) are valid for all $s \leqslant T$. This is impossible, however, since these equations imply the relation

$$B_2 \sum_{j=0}^{\alpha-1} B_0^{j}\, y(\tau + jh,T)\, l_0' \equiv 0, \qquad 0 \leqslant \tau \leqslant h$$

which contradicts condition (9). Theorem 3 has been proved.

Note 2. By virtue of certain results of [2] for $B_0 \equiv 0$ and $0 < T \leqslant h$ the requirements of Theorem 3 are not only necessary, but also the sufficient conditions of solvability of Problem 1 for Eq. (10).

Finally, let us cite certain solvability conditions for Problem 1 in the case of one-dimensional equations with constant coefficients,

$$x^{\cdot}(t) = \sum_{i=1}^{m} b_i x(t - h_i) + ax(t) + b_0 x^{\cdot}(t - h), \qquad t > 0$$

Theorem 4. If $b_i > 0,\ h_i > 0$, $i = 1, \ldots, m$, and if the constants $h > 0$, $b_0 \geqslant 0,\ ab_0 \geqslant 0$, Problem 1 for Eq. (24) is solvable for all $x_0,\ x_1$.

Proof. Since the coefficients of Eq. (24) are constant, the fundamental solution $y(t - s)$ of this equation, which depends solely on the difference between the arguments, is defined by the relations

$$y^{\cdot}(t) = \sum_{i=1}^{m}\sum_{j=0}^{\alpha-1} b_i b_0^{j} y(t - h_i - jh) + \sum_{i=0}^{\alpha-1} ab_0^{i} y(t - ih) \qquad 0 < t \leqslant T \tag{25}$$

$$y(0) = 1, \qquad y(t) = 0, \quad t < 0$$

By virtue of Lemma 1 we need merely establish the positiveness of the function $y(t)$ for all $0 < t \leqslant T$. To this end we denote one-dimensional Wiener processes mutually independent for all distinct values of the indices by $\xi_{ij}(t)$ and define the process $z(t)$

by means of Ito's stochastic differential equation with lag [3]:

$$dz(t) = \frac{1}{2} az(t)\,dt + \sum_{i=1}^{m}\sum_{j=0}^{\alpha-1} \sqrt{b_i b_0^j}\, z(t - h_i - jh)\, d\xi_{ij}(t) + \sum_{i=1}^{\alpha-1} \sqrt{a b_0^i}\, z(t - ih)\, d\xi_{0i}(t), \quad 0 < t \leqslant T \tag{26}$$

under the initial conditions

$$z(0) = 1, \quad z(t) = 0, \quad t < 0 \tag{27}$$

As in the proof of Theorem 1 in [3], we can verify the fact that the solution $z(t)$ of problem (26), (27) for certain constants c_1, c_2 satisfies the estimate

$$Mz^4(t) \leqslant c_1 e^{c_2 t} \tag{28}$$

(M represents the mathematical expectation) from which, on the basis of Ito's formula ([4], p. 506), we infer that $y(t) = Mz^2(t)$. But $Mz^2(t) \geqslant [Mz(t)]^2$, so that to prove Theorem 4 we need merely verify that $r(t) = Mz(t) > 0$ for $0 \leqslant t \leqslant T$. By virtue of Ito's formula and relations (26)–(28) we find that the function $r(t)$ is the solution of the ordinary differential equation

$$r^{\cdot}(t) = 1/2\, ar(t), \quad 0 < t \leqslant T$$

under the initial condition $r(0) = 1$. Hence, $r(t) > 0$ for all $0 \leqslant t \leqslant T$. Theorem 5 has been proved.

Note 3. Repeating verbatim the argument used to prove Theorems 2–4, we see that they are also valid for inhomogeneous equations of the form (10).

Note 4. Some of the results of the present study can be readily extended to equations with variable coefficients.

Example 1. Let us consider the one-dimensional equation

$$x^{\cdot}(t) = x(t - h) + x^{\cdot}(t - 2h), \quad t > 0,\ h > 0$$

Let the number $x_0 = 1$, $x_1 = 2 + h$, the initial function $\psi(\theta) \equiv 0$, and the instant $T = 2h$. Then $y(\theta + h, 2h) = 1 - \theta$ for $-h \leqslant \theta \leqslant 0$ and the constant $\beta = 1$ (see formula (6)). From this and from (9) we infer that $\rho^2_0 = h + h^2 + {}^1/_3\, h^3 > 0$. Hence, the optimal resolvent $\varphi(\theta)$, which must satisfy the conditions

$$\int_{-h}^{0} \varphi^2(s)\,ds = \rho_0^{-2}, \qquad \int_{-h}^{0} \varphi(s)\, y(s + h, 2h)\,ds = 1$$

is equal to

$$\varphi(\theta) = \rho_0^{-2}(1 - \theta), \quad -h \leqslant \theta \leqslant 0$$

Example 2. The one-dimensional equation $x^{\cdot}(t) = x(t) + x^{\cdot}(t - h) + x(t - h)$, $t > 0$, $h > 0$ is given.

Let us suppose that $x_0 = 1$, $x_1 = 2e^h$, that the constant $T = h$, and that the intial function $\psi(\theta) = 1$, $(-h \leqslant \theta \leqslant 0)$. It is easy to show that $y(\theta + h, h) = e^{-\theta}$. In accordance with Items (1) and (2) of Lemma 1, the optimal initial function $\varphi(\theta)$ is equal to $\varphi(\theta) = \rho_\theta^{-2} e^{-\theta}$, since the constant $\rho = 1$ and $2\rho^2_0 = e^{2h} - 1$.

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THE TIME-OPTIMAL CONTROL PROBLEM IN SYSTEMS WITH CONTROLLING FORCES OF BOUNDED MAGNITUDE AND IMPULSE

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The problem of time-optimal control is considered in the case where the controlling forces are bounded in magnitude and in impulse at the same time. The study is carried out with the aid of attainability domains. The case where the boundaries of these domains have plane portions and corners is considered. The problem of optimal control synthesis is solved for certain second-order systems with the indicated restrictions imposed on the controlling forces.

1. Statement of the problem. Let us consider the control system described by the following linear matrix differential equation with real constant coefficients:

$$dx/dt = Ax + Bu \tag{1.1}$$

Here $x = \| x_i \|$, $A = \| a_{ij} \|$, $B = \| b_{is} \|$, $u = \| u_s \|$ are matrices of order $(n \times 1)$, $(n \times n)$, $(n \times r)$, $(r \times 1)$, respectively, and $u_s = u_s(t)$ is a measurable function of time which satisfies the following restrictions simultaneously:

$$|u_s(t)| \leqslant M_s \qquad (M_s = \text{const} > 0) \tag{1.2}$$

$$\int_0^\infty |u_s(\tau)|\, d\tau \leqslant C_s^\circ \qquad (C_s^\circ = \text{const} > 0) \tag{1.3}$$

By b_s $(s = 1, \ldots, r)$ we denote the sth column of the matrix B ($b_s \neq 0$ for all $s = 1, \ldots, r$). Condition (1.2) expresses the boundedness of the controlling force, and condition (1.3) expresses (from the physical standpoint) the boundedness of the impulse of the controlling force. Inequality (1.3) in certain cases represents the limitation of the propellant capacity of a thruster.

We shall consider the problem of bringing system (1.1) to the origin in the minimum time by means of a control which satisfies conditions (1.2), (1.3) (e. g. see [1], and, among other things, the problem of synthesizing the time-optimal control.

When restriction (1.2) alone is imposed, the time-optimal control is, as we know [2-5], a relay control (we denote the minimum time in this case by $\theta = \theta(x)$). The problem of synthesizing such a control consists in splitting the space X_n composed of the phase coordinates $x_1, \ldots, x_n$ by the switching surfaces into domains in which the controls $u_s(t)$ assume the values M_s and $-M_s$ $(s = 1, \ldots, r)$. Once this splitting has been effected, the optimal control is known as a function of the phase coordinates